HIGHER SPIN PROBLEM IN FIELD THEORY

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Aims

- Brief non-expert non-technical review of some old and current problems of higher spin field theory
- Specific aspects of Lagrangian construction for higher spin fields

- Higher spin fields
- Motivations
- Problem of auxiliary fields. Spin 2 and 3 examples
- Problem of unconstrained formulation
- Problem of coupling to external fields
- Problem of causality
- Problem of higher spin interactions
- Relation of higher spin field theory and string theory
- BRST approach to higher spin field Lagrangian construction

Notion of spin is a direct consequence of special relativity symmetry: physical laws are invariant under the space-time translations and Lorentz rotations

- Space-time translations

$$x'^{\mu} = x^{\mu} + a^{\mu}$$

  x^{μ} are the space-time coordinates and a^{μ} is an arbitrary constant four-vector. $\mu, \nu = 0, 1, 2, 3.$

- Lorentz transformations

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$$\Lambda^{T} \eta \Lambda = \eta$$

  $\eta = (\eta_{\mu\nu})$ is Minkowski metric.

Poincare group

- Set of transformations (a, Λ) forms a group (Poincare group).
- Generators $P_{\mu}, J_{\mu\nu}$

Fields with given spin

- Casimir operators $C_1 = P^2, C_2 = W^2, (W_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}P^\beta J^{\gamma\delta})$
- Irreducible representations of the Poincare group are characterized by eigenvalues of the Casimir operators.
  Massive case. The eigenvalues are m^2 (for P^2) and $-m^2 s(s+1)$ (for W^2)
  m is a mass and s is a spin, $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...$
  Massless case $m = 0$, $W_\mu = \pm\lambda P_\mu$,
  $\lambda = 0, \frac{1}{2}, 1, \frac{3}{2}, 2...$
  λ is a helicity (spin of massless particle)
- Irreducible representations of the Poincare group can be realized on space-time fields
  Bosonic field with mass m and spin $s = n$, $\phi^{(s)}(x) = \phi_{\mu_1\mu_2\ldots\mu_n}(x)$:
  $\phi_{\mu_1\mu_2\ldots\mu_n} = \phi_{(\mu_1\mu_2\ldots\mu_n)}$
  ${\phi^{\mu_1}}_{\mu_1\mu_3\ldots\mu_n} = 0$
  $\partial^{\mu_1}\phi_{\mu_1\mu_2\ldots\mu_n} = 0$
  $(\partial^2 + m^2)\phi_{\mu_1\mu_2\ldots\mu_n} = 0$

Fields with given spin

Fermionic field with given mass m and spin $s = n + \frac{1}{2}$, $\psi^{(s)} = \psi_{\mu_1\mu_2\ldots\mu_n}$:

$\psi_{\mu_1\mu_2\ldots\mu_n} = \psi_{(\mu_1\mu_2\ldots\mu_n)}$

$\gamma^{\mu_1}\psi_{\mu_1\mu_2\ldots\mu_n}$

$\partial^{\mu_1}\psi_{\mu_1\mu_2\ldots\mu_n} = 0$

$(i\gamma^{\mu}\partial_{\mu} - m)\psi_{\mu_1\mu_2\ldots\mu_n} = 0$

Dirac tensor spinor, $\psi_{\mu_1\ldots\mu_n} = \psi_{\mu_1\ldots\mu_n,a}, a = 1,2,3,4.$

Terminology:

Lower spin fields: $s = 0, \frac{1}{2}, 1$

Higher spin fields: $s = \frac{3}{2}, 2...$

Central problem of higher spin field theory:

CONSTRUCTION OF LAGRANGIAN FORMULATION FOR ARBITRARY HIGHER SPIN FIELDS INTERACTING AMONG THEMSELVES, OR/AND LOWER SPIN FIELDS OR/AND EXTERNAL FIELDS

Problem of classical field theory

- Lagrangian formulation for free higher spin fields is constructed although some aspects are still under investigation
- Inserting the naive interacting terms into free field Lagrangian yields the physical or/and mathematical contradictions: inconsistence of equations of motion, faster then light propagations
- Some cases of higher spin Lagrangian formulation are known:
  Interacting massless spin 2 field theory – general relativity. General covariance
  Interacting massless spin 2 and $\frac{3}{2}$ fields theory – supergravity. Local supersymmetry
- Higher spin (super)symmetry?

Lower spin field Lagrangians

- real spin-0 field:

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2) - V(\phi)$$

  Arbitrary potential $V(\phi)$.

- massless spin-1 field:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + f(F^{\mu\nu}F_{\mu\nu})$$

  Strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, arbitrary function $f(F^{\mu\nu}F_{\mu\nu})$ of gauge invariant combination $F^{\mu\nu}F_{\mu\nu}$.

- spin-$\frac{1}{2}$ field coupled to massless spin-1 field

$$\mathcal{L} = \bar{\psi}(\gamma^\mu \mathcal{D}_\mu - m)\psi + f(\bar{\psi}\psi)$$

  Arbitrary function $f(\bar{\psi}\psi)$, covariant derivative $\mathcal{D}_\mu = \partial_\mu - ieA_\mu$

- general relativity:
  Lagrangian is very restrictive by general covariance.
- supergravity:
  Lagrangian is very restrictive by local supersymmetry.
- interacting higher spin field Lagrangian:
  General Lagrangian construction is unknown.

Why Lagrangian formulation for higher spin fields

- New very non-trivial models of classical field theory.
- Lagrangian formalism is a basis for quantum higher spin field theory.
- New possibilities to unification of fundamental interactions.
  Quantum gravity?
- Possibilities for new phenomenology beyond the standard model.
  Identification of dark matter in cosmology with massive higher spin particles.
- Some composite massive higher spin particles (hadron resonances) are experimentally detected. They really propagate in gravitational field and therefore one can hope that there exist the Lagrangian formulation for their interactions with gravitational field.
- Superstring theory predicts an infinite number of massive higher spin particles. In principle they can survive in low energy limit. Therefore one can hope that there exist a way to describe these particles on the base of field theory.
- AdS/CFT correspondence.

STATEMENT:
Let $\phi^{(s)}$ is a field with mass m and spin s. Lagrangian, describing a free dynamics of the field $\phi^{(s)}, s > 1$ can not be constructed only in terms of this field. One should take into account the non-dynamical fields $\phi^{(s')}$ with $s' < s$ as well.
(Fierz and Pauli).
Non-dynamical fields in Lagrangian – auxiliary fields.
Problem of auxiliary fields:
Which fields $\phi^{(s')}, s' < s$ and in which form should be written in the Lagrangian for basic field $\phi^{(s)}$ in order to get consistent dynamics for basic field?

Why one can expect a problem for higher spin Lagrangian?
Field $\phi^{(s)}$ with given mass m and spin $s = n$ is defined by the relations:

$$\phi_{\mu_1\mu_2\ldots\mu_n} = \phi_{(\mu_1\mu_2\ldots\mu_n)}$$
$$\phi^{\mu_1}{}_{\mu_1\mu_3\ldots\mu_n} = 0$$
$$\partial^{\mu_1}\phi_{\mu_1\mu_2\ldots\mu_n} = 0$$
$$(\partial^2 + m^2)\phi_{\mu_1\mu_2\ldots\mu_n} = 0$$

Let $\mathcal{L}$ is the Lagrangian, depending only on $\phi^{(s)}$, and $S[\phi^{(s)}]$ is the corresponding action.

Equations of motions:

$$\frac{\delta S[\phi^{(s)}]}{\delta \phi^{(s)}} = 0$$

Equations of motion must be compatible with the relations, defying the field $\phi^{(s)}$, i.e. they must reproduce the constraints on the field $\phi^{(s)}$, which define the irreducible representations of the Poincare group. That means that the Lagrangian must have a very special structure. One can not say a priori that such a Lagrangian actually exists.

General solution for free theories (Singh and Hagen for massive case, Fronsdal for massless case):

- True Lagrangian depends not only on basic field with spin s but also on non-propagating fields with spin less s (auxiliary fields).
- Eliminating the auxiliary fields from the equations of motion yield correct constraints for basic field.

Generalization for higher spin fields in AdS space.

Problem of auxiliary fields

- Massive bosonic spin s field.
  Lagrangian is given in terms of totally symmetric traceless tensor fields with spins $s, s-2, ..., 1, 0$.
- Massive fermionic spin s field.
  Lagrangian is given in terms of totally symmetric γ – traceless tensor-spinor fields with spins $s, s-\frac{3}{2}, s-\frac{5}{2}, ..., \frac{1}{2}$.
- Massless bosonic spin s field.
  Lagrangian is given in terms totally symmetric traceless tensor fields with spins $s, s-2$.
  It can be imbedded into a single double traceless spin s field.
- Massless fermionic spin s field.
  Lagrangian is given in terms of totally symmetric γ – traceless tensor-spinor fields with spins $s, s-2$.
  It can be imbedded into a single double γ – traceless tensor-spinor field.
- In all cases the Lagrangians are given in terms of constrained fields.
- Problem of propagating auxiliary fields in interacting theory.

Example: massive spin 2 field

Massive spin 2 field is described by the symmetric rank 2 tensor field $\phi_{\mu\nu}$ under the constraints:
$(\partial^2 + m^2)\phi_{\mu\nu} = 0, \phi^\mu{}_\mu = 0, \partial^\mu \phi_{\mu\nu} = 0$

Lagrangian contains the basic traceless field $\phi_{\mu\nu}$ and auxiliary field ϕ and has the form

$$\mathcal{L} = \frac{1}{2}\partial^\alpha \phi^{\mu\nu} \partial_\alpha \phi_{\mu\nu} - \frac{1}{2} m^2 \phi^{\mu\nu}\phi_{\mu\nu} - \partial^\alpha \phi_{\alpha\mu} \partial^\beta \phi_\beta{}^\mu + \frac{2}{3}\partial^\mu \phi \partial^\nu \phi_{\mu\nu} + \frac{1}{3}\phi(\partial^2 + 2m^2)\phi$$

Consequences of the equations of motion: $\phi = 0$ and $\phi_{\mu\nu}$ satisfies the true constraints.
However: If to put the auxiliary field ϕ=0 in the Lagrangian, the corresponding equations of motion do not reproduce the true constraints.

New field $h_{\mu\nu} = \phi_{\mu\nu} + \frac{1}{3}\eta_{\mu\nu}\phi$ and Lagrangian
$\mathcal{L} = \frac{1}{2}\partial^\alpha h^{\mu\nu}\partial_\alpha h_{\mu\nu} - \frac{1}{2}m^2 h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}\partial^\alpha h^\mu{}_\mu \partial_\alpha h^\nu{}_\nu + \frac{1}{2}m^2 h^\mu{}_\mu h^\nu{}_\nu -$
$\partial^\alpha h_{\alpha\mu}\partial^\beta h_\beta{}^\mu + \partial^\mu h^\nu{}_\nu \partial^\alpha h_{\alpha\mu}$
m=0, linearized Lagrangian of general relativity.

Example: massive spin 3 field

Massive spin 3 field is described by the totally symmetric rank 3 tensor field $\phi_{\mu_1\mu_2\mu_3}$ under the constraints:

$(\partial^2+m^2)\phi_{\mu_1\mu_2\mu_3}=0, \phi^{\mu}{}_{\mu\nu}=0, \partial^{\mu_1}\phi_{\mu_1\mu_2\mu_3}=0$

Lagrangian contains the basic traceless field $\phi_{\mu_1\mu_2\mu_3}$ and the auxiliary fields ϕ_μ, ϕ

$$\mathcal{L}=-\frac{1}{2}\phi^{\alpha\beta\gamma}(\partial^2+m^2)\phi_{\alpha\beta\gamma}-\frac{3}{2}\partial^\gamma\phi_{\alpha\beta\gamma}\partial^\delta\phi_\delta{}^{\alpha\beta}-\frac{12}{5}\phi^\alpha\partial^{\beta\gamma}\phi_{\alpha\beta\gamma}+$$
$$+\frac{6}{5}\phi^\alpha(\partial^2+m^2)\phi_\alpha-\frac{6}{25}\partial^\alpha\phi_\alpha\partial^\beta\phi_\beta-\frac{18}{25}m\phi_\alpha\partial^\alpha\phi+\frac{9}{25}\phi(\partial^2+4m^2)\phi$$

Consequences of the equations of motion: $\phi_\alpha=0, \phi=0$ and basic field $\phi_{\alpha\beta\gamma}$ satisfies the true constraints.

However: If to put $\phi_\alpha=0, \phi=0$ in the Lagrangian from the very beginning, the corresponding equations of motion do not reproduce the true constraints.

The coefficients at the terms with auxiliary fields can be changed by scale transformations of auxiliary fields, $\phi'_\alpha=c_1\phi_\alpha, \phi'=c_2\phi$. General property of higher spin field Lagrangian.

Problem unconstrained formulation

Higher spin Lagrangians are written in terms of constrained fields

- Massive bosonic theory: traceless fields
- Massive fermionic theory: γ–traceless fields
- Massless bosonic theory: double traceless fields
- Massless fermionic theory: double γ–traceless fields

Problem of unconstrained formulation: finding the equivalent Lagrangians in terms of unconstrained fields.

Motivations

- Convenience of calculating the variational derivatives of action
- More simple form of the Lagrangians for massive theories
- Links with tensionless limit of superstring theory
- Possible way to interacting theory

Problem unconstrained formulation

General idea: new auxiliary fields

Solution (2007, 2008)

- Universal set of totally unconstrained fields.
- Universal (local) Lagrangian (without higher derivatives).
- Gauge invariant formulation (even for massive theory).
- Elimination of the auxiliary fields from the equations of motion reproduces the Singh-Hagen and Fronsdal formulations.

Rule of minimal inclusion of interaction with external fields (electromagnetic field, Yang-Mills field, gravitational field) in lower spin theories:
Replacement in free theory the ordinary partial derivative ∂_μ by covariant derivative $\mathcal{D}_\mu = \partial_\mu - \Gamma_\mu$ with some connection Γ_μ.
Following this rule ones replace in the constraints and Lagrangian the partial derivatives by covariant.
However:

- Partial derivatives commute, $[\partial_\mu, \partial_\nu] = 0$
- Covariant derivatives do not commute, $[\mathcal{D}_\mu, \mathcal{D}_\nu] \sim \mathcal{R}_{\mu\nu}$. Curvature (strength) tensor $\mathcal{R}_{\mu\nu} \neq 0$.

In process of eliminating the auxiliary fields from equations of motion in free theory we commute the partial derivatives. Since the covariant derivative do not commute, we get some extra terms depending on curvatures, which are not canceled in general. Equations of motion are inconsistent. Minimal inclusion of interaction does not work! Non-minimal interaction?
General solution is unknown. May be it does not exist.
The only known external background for arbitrary spin field is a constant curvature space. Partial cases: spin $\frac{3}{2}$ and 2 fields in Einstein space, massive spin 2 field in constant electromagnetic field.

Another aspect of inconsistency.
Velo-Zwanziger problem: velocity of propagating the spin $\frac{3}{2}, 2$ fields minimally coupled to constant electromagnetic field.
Method:
Let there is a system of second order partial differential equations
$A^I{}_J{}^{\mu\nu}\partial_\mu\partial_\nu\Phi_J + ... = 0$
Characteristic equation
$\mathrm{D}(n) = \det(A^I{}_J{}^{\mu\nu}n_\mu n_\nu) = 0$
The equation is called hyperbolic if there exist a real four-vector n_μ satisfying the equation $\mathrm{D}(n) = 0$. Maximal velocity of signal propagation is $\frac{n_0}{|\vec{n}|}$.

- All massive fields with spin s have a number of components grater then $2s + 1$, it means that the equations of motion contains the non-dynamical constraints.
- Before computing the characteristic determinant ones need to take into account the constraints. After that $\mathrm{D}(n)$ depends on field strength $F_{\mu\nu}$
- There are the values of $F_{\mu\nu}$ where $\frac{n_0}{|\vec{n}|} > c$. Violation of causality.
- Solution is unknown

However:

- String theory allows to derive completely consistent and causal Lagrangian for massive spin 2 field coupled to constant electromagnetic field in d=26. Coupling to external field is strongly non-minimal (modification of kinetic term in Lagrangian by external field depending terms). Absolutely unclear how such a Lagrangian can be obtained in field theory.
- The best what we have up to now are the attempts to derive the linear in coupling to electromagnetic field vertex, i.e. vertex of the form $\partial\phi^{(s)}\partial\phi^{(s)}F$. It is valid up to the terms of F^2. Perhaps, if to sum all possible vertices with all powers of F ones get the result of string theory. Such a construction demands to deform not only the Lagrangian but also the gauge transformations. Systematic procedure is not developed yet even in linear approximation.

Higher spin propagation in AdS space is consistent and causal.
Consistent formulation for scalar field coupled to external fields of all integer spins.

Problem: Lagrangian for interacting dynamical higher spin fields.

- Reliable result is obtained only for cubic vertex of the form $\partial\phi^{(s)}\partial\phi^{(s')}\partial\phi^{(s'')}$ with some s, s', s''.
- Consistent Lagrangian up to the terms of order ϕ^4.
- Consistent gauge transformation up to the terms of order ϕ^2
- Coupling constant of negative mass dimension.
- Lessons: consistent interacting Lagrangian and gauge transformations should include the contributions of all powers of higher spin fields up to infinity. Terms with higher derivatives in Lagrangian and gauge transformations.
- Fradkin-Vasiliev construction for cubic vertex:
  Cubic interaction of all bosonic and fermionic higher spin field with gravity.
  New type of symmetry mixing infinite number of fields with different spins.
  Consistent in given (cubic) order. Consistent only on AdS space.
- Vasiliev consistent equation of motion for all interacting higher spin field. Problem of Lagrangian.

General lesson:

- True completely consistent Lagrangian must include infinite number of vertices and contributions from all integer and half integer spins $s = \frac{3}{2}, ...$
- Higher derivatives in interaction vertices and gauge transformations.
- Coupling constant of negative mass dimension.
- Typical properties of effective theory.

Natural question: Which fundamental theory can yield to such effective theory?
The only candidate at present: superstring theory.

- Fundamental elements of Nature are not the point-like particles but the little curves – strings.
- Strings can vibrate. Each vibrational mode looks like a particle with given mass and spin.
- String spectrum includes a finite number of massless particles and infinite number of massive ones.
- Interaction of strings means interaction of all particles from their spectra.
- At ultra high energies one can neglect all masses and we should get some kind of effective consistent gauge theory containing the interacting massless fields with all integer and half integer spins. Problem is open.

Motivation. Development of the universal Lagrangian formulation for higher spin fields which:

- automatically includes the auxiliary fields
- automatically is unconstrained
- automatically leads to gauge invariant Lagrangian for massive fields
- reproduces all known results
- open a new way to construction of higher spin interactions

One of the possible approaches to higher spin field Lagrangians – BRST(Becci, Rue, Stora, Tyutin) - BFV(Batalin, Fradkin, Vilkovisky) construction:
Method of covariant canonical quantization of gauge theories

- Gauge theories are characterized by first class constraints T_a in phase space
  $\{T_a, T_b\} = f^c{}_{ab} T_c$
- Nilpotent BRST – charge
  $Q = \eta^a T_a + \frac{1}{2}\eta^a \eta^b f^c{}_{ab}\mathcal{P}_c + ...$
  $\eta^a, \mathcal{P}_b$ – canonically conjugate ghost variables.
  $\{Q, Q\} = 0$
- After quantization:
  Q – nilpotent Hermitian operator, acting in enlarged space of states
- Physical subspace:
  $Q|\text{phys} >= 0$
- The physical states are defined up to transformation
  $|\text{phys}' >= |\text{phys} > +Q|\Lambda >$
- S-matrix is unitary is physical subspace
- Initial point: classical Lagrangian formulation.
  Unclear how to apply to higher spin field theory where namely classical Lagrangian formulation is a main problem.

General scheme of application of BRST – construction in higher spin field theory:

- Realization of conditions on fields as the operators acting in auxiliary Fock space and treatment of these operators as first class constraints (in terms of commutators) of some unknown yet Lagrangian theory.
  Quantum technique in classical field theory.
- If an algebra of such operators is unclosed ones add some new operators while the algebra of whole set of operators will be closed.
- To construct Hermitian BRST - charge ones should use the operators which can not be treated as the constraints.
- In general one can expect that the algebra under consideration is non-linear. For example, in AdS space such an algebra schematically looks like $[T, T] \sim T + T^2$.
- Construction of BRST-charge on the base of above algebra.
  In case of non-linear algebra construction of BRST-charge is a separate problem.
- Extension of the Fock space by vectors $|\Phi>$ depending on ghost variables. Postulation of the equation $Q|\Phi>= 0$, which is treated as the higher spin field equation of motion.

BRST approach to higher spin Lagrangian construction

- Proof that the equation $Q|\Phi>=0$ reproduces the conditions on fields we begin with. For example, the equations which determine the irreducible representations of Poincare or AdS groups are consequences of the equation $Q|\Phi>=0$.
- Since the equation $Q|\Phi>=0$ is invariant under transformation $|\Phi'>=|\Phi>+Q|\Lambda>$ ones get automatically gauge invariant equations of motion.
- Construction of Lagrangian leading to the equation of motion $Q|\Phi>=0$. For example, for bosonic fields such a Lagrangian has a simple form $\mathcal{L}\sim<\Phi|KQ|\Phi>$.
  $<\Phi_1|\Phi_2>$ is inner product in enlarged Fock space, operator K depends only on ghost variables.
  Lagrangian is completely written in terms of BRST-charge.
- For construction of interacting Lagrangians ones need to develop a deformations of BRST-charge of free theories preserving nilpotency (at least approximately).

Programme is partially realized.

Summary of results obtained

- Lagrangian formulation is derived for totally symmetric massive and massless, bosonic and fermionic fields and for fields with mixed symmetry of indices in flat and AdS spaces.
- The auxiliary fields have the natural origin as the coefficients at ghost operators in vector $|\Phi >$.
- Formulation is completely unconstrained. No any trace conditions are imposed from the very beginning, all conditions are the consequences of higher spin equation of motion $Q|\Phi >= 0$.
- Formulation is gauge invariant even for massive fields.
- Some partial cases of cubic higher spin massless fields are found.

Open problems of the BRST-approach

- General structure of higher spin interaction vertices in terms of BRST-approach
- Supersymmetric higher spin Lagrangian formulation in terms of BRST-approach
- Quantum effects in terms of BRST-approach

THANK YOU VERY MUCH